On the direction of the momentum flux in an unstable flow with mixed baroclinic and barotropic shear: A linear stability analysis

Madeleine K. Youngs and Edwin P. Gerber

Center for Atmosphere Ocean Science, Courant Institute of Mathematical Sciences, New York University

Corresponding author: Madeleine K. Youngs, my2368@nyu.edu
ABSTRACT: Eddy-mean flow interactions in baroclinically unstable systems are critical for setting the mean state of the ocean and atmosphere. We do not fully understand, however, what processes determine the direction of the eddy momentum or buoyancy fluxes in flows when there is both horizontal and vertical shear, and thus potential for both baroclinic and barotropic instability. This study illustrates the non-separability of horizontal and vertical fluxes in a mixed shear system using a linear stability analysis of a two-layer quasi-geostrophic channel model with parameter sweeps of stratification, planetary vorticity gradient, and jet width. In a first experiment, a Gaussian jet in the upper layer, representing a generic oceanic or atmospheric jet that is primarily baroclinically unstable is considered. The relative narrowness of the jet is varied to introduce barotropic instability and probe when the momentum fluxes switch from upgradient to downgradient, a signature of barotropic instability. It is shown that a reversal of the meridional potential vorticity gradient (the necessary condition for barotropic instability) is neither a necessary nor sufficient condition for a reversal of the momentum fluxes. A second experiment utilizes a cosine jet and topography, effectively exchanging the horizontal and vertical gradients in the system and highlighting the interconnection between momentum and buoyancy fluxes through the potential vorticity. The importance of a momentum flux closure for mixed barotropically-baroclinically sheared flows for current oceanographic research is discussed and specific regions are spotlighted where the dynamics merit further investigation.
1. Introduction

Barotropic and baroclinic instability play an essential role in the large scale dynamics of the ocean and atmosphere. Baroclinic instability generate storms in the atmosphere, effecting the poleward transport of energy through the midlatitudes and driving the surface westerlies (e.g., Vallis 2006, and ref. therein), and eddies in the ocean, setting the stratification of the global ocean (Marshall and Speer 2012). In all, these instabilities are critical for setting the mean state of our climate system (Lorenz 1967). In the ocean, barotropic and baroclinic instabilities cannot be sufficiently resolved in current generation coupled climate models, and are thus parameterized, most often as a diffusion of buoyancy along isopycnals. This decreases the baroclinic shear, but does not account for the momentum fluxes associated with both instabilities and their effects on the mean flow (Gent and McWilliams 1990). As a result, the flow in models suffers from biases in the mean state and variability in many regions (e.g., Hewitt et al. 2020).

The necessary conditions for barotropic and baroclinic instability require that there be a sign change of the potential vorticity (PV) gradient somewhere in the domain (Charney and Stern 1962). Purely barotropic instability requires a sign change of the potential vorticity gradient in the horizontal (within a layer), permitting a downgradient momentum flux to extract energy from the mean kinetic energy of the flow (Rayleigh 1880; Drazin and Howard 1966). For a purely baroclinic instability, there must be a sign change in the potential vorticity gradient between the layers (Charney 1947; Eady 1949). In the real world, however, flows almost always exhibit both vertical and horizontal shear, and thus present two mean energy reservoirs: kinetic and potential energy. The necessary conditions for instability in this case do not indicate which energy reserves could be tapped, but just that energy could be extracted from the mean.

In a baroclinically unstable fluid with a horizontal shear, interesting interactions occur between the instability and the horizontal shear. There can be upgradient momentum fluxes associated with the baroclinic instability when there is a weak horizontal shear (Pedlosky 1964), but also downgradient momentum fluxes typically associated with barotropic instability with a strong horizontal shear. Early work by McIntyre (1970) developed analytic solutions to the Eady (1949) model with linear barotropic shear, establishing conditions ensuring that the momentum fluxes remained upgradient. Held (1975) found that when a two-layer flow has the same sign potential vorticity gradient in both layers, the direction of the momentum fluxes is determined.
Renewed interest in the impact of meridional shear arose in the context of the barotropic governor mechanism of James and Gray (1986), who observed that, seemingly paradoxically, reducing friction at the surface leads to a reduction of eddy activity. They attribute the decay in eddies to the development of strong barotropic jet, whose shear inhibited baroclinic instability. Indeed, James (1987) showed how the introduction of linear barotropic shear in a two layer quasi-geostrophic model reduced the baroclinic growth rates. Building on this work, Nakamura (1993a) developed an analytic model for instability in the context of both vertical and horizontal shear, showing both the reduction in baroclinic growth rates in the presence of shear and its impact on the momentum fluxes.

The momentum fluxes associated with the nonlinear phase of baroclinic instability have justifiably received attention as well, beginning with the “eddy life cycle” experiments of Simmons and Hoskins (1978). Here barotropic shear has been shown to have a significant impact on the momentum fluxes, (e.g., Simmons and Hoskins (1980), Nakamura (1993b), and Thorncroft et al. (1993); see Maher et al. (2019) for a recent review). Work in an atmospheric context, however, has largely focussed on cases where barotropic shear impedes baroclinic instability, but the flow itself remains barotropically stable. This reflects the fact that the large scale forcing of the atmospheric is baroclinic, driven by the meridional gradiant in insolation: barotropic shear must be generated by upgradient momentum fluxes from baroclinic instability, this process feeding back on itself in the barotropic governor. In the ocean, however, mechanical forcing by the surface winds opens up wider room for the generation of jets that can be barotropically or baroclinically unstable, or both (e.g., Gula et al. 2015; Semtner and Mintz 1977; Youngs et al. 2017; Zhai et al. 2008). One other consideration that distinguishes the atmosphere and the ocean case is the scale difference between Rossby waves and eddies, which are on the same scale in the atmosphere, but eddies are significantly smaller in the ocean (Williams et al. 2007).

The linear problem for mixed baroclinic/barotropic instability in this oceanic context remains less well explored. Early work by Killworth (1980) examined the parameter space of instabilities with horizontal and vertical shear and found that when the horizontal length scale of the shear was larger than the internal deformation radius, baroclinic conversion dominated, but when the horizontal length scale was much smaller than the deformation radius, then barotropic conversion dominated. Holland and Haidvogel (1980) obtained similar results in a more explicitly oceanic context. The
modest aim of this manuscript is to explore the linear stability of flow in a zonal channel over a wider parameter space, assessing the direction of the momentum and buoyancy fluxes associated with infinitesimal perturbations. While it is ultimately the fully nonlinear problem that must be addressed, we find considerable complexity even in this simpler, linear context.

We first consider a case of relevance to the Southern Ocean in particular. After describing our model and diagnostics in Section 2, a Gaussian-shaped baroclinic jet is explored in Section 3. Increasing the shear in the upper layer allows us to introduce and vary the strength of barotropic instability in the system. We generate a wide parameter sweep across the system, probing how changes in vertical and horizontal shear impact the momentum and buoyancy fluxes associated with the linear instability.

To illustrate the symmetry between barotropic and baroclinic instability – and hence the degree to which they are intertwined – we then construct a second jet configuration that allows us to effectively rotate the system in Section 4: a barotropically unstable cosine-shaped jet with baroclinicity introduce through interaction with bottom topography. A parameter sweep with this jet further illustrates the complexity of fluxes, even in the linear limit. We discuss and conclude in section 5, providing context and motivation for why this open question is relevant for continued consideration in the present day ocean.

2. Model

In order to address the question of what sets the direction of the momentum fluxes in linear mixed shear instability, we consider a two-layer quasi-geostrophic channel model. This is the simplest model that represents the dynamics of an atmosphere or ocean jet with both horizontal and vertical shear. The two-layer quasigeostrophic linear problem is a well studied configuration, but we put our own twist on it by using the basic state and the topography to create specific meridional PV gradients and the relationships between and across the layers. In this section we describe the model and derive the linear stability problem, present diagnostics used, and discuss the eigenvalue solver used to solve the linear stability problem.
The Linear Stability Problem

The derivation that follows parallels Pedlosky (1987). We consider the two–layer quasi–
geostrophic potential vorticity (QGPV) equations on a $\beta$-plane in a non-dimensional form:

\[
\frac{\partial}{\partial t} + \frac{\partial \psi_n}{\partial x} \frac{\partial}{\partial y} - \frac{\partial \psi_n}{\partial y} \frac{\partial}{\partial x} \left[ \frac{\partial^2 \psi_n}{\partial y^2} - F_n(-1)^n (\psi_2 - \psi_1) + \eta_b \delta_{n2} \right] = 0 \quad n = 1, 2 \quad (1)
\]

where $\psi_n$ is the stream function, $\delta_{ij}$ is the Kronecker delta function and

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.
\]

The non-dimensional parameter

\[
\beta = \beta_0 \frac{L^2}{U}, \quad (2)
\]

where $\beta_0$ is the meridional gradient in planetary vorticity, $L$ is a length scale given by the width of the channel, and $U$ is the velocity scale of the jet, represents the planetary vorticity gradient.

Additional non-dimensional parameters,

\[
F_n = \frac{f_0^2 L^2}{g(\rho_2 - \rho_1) / \rho_0 D_n}, \quad (3)
\]

where $f_0$ is the planetary vorticity, $\rho_n$ is the density of the $n$-th layer, $\rho_0$ is a reference density and $D_n$ is the depth of the layer, measure the effect of each layer on the other layer. We introduce the general stratification parameter:

\[
F = \frac{f_0^2 L^2}{g(\rho_2 - \rho_1) / \rho_0 D}, \quad (4)
\]

where $D$ is the total depth. $F_n$ can be related to $F$ by a ratio of depth of top layer to lower layer $\Delta = D_1 / D_2$: $F_1 = F (1 + \Delta) / \Delta$ and $F_2 = F (1 + \Delta)$. For all of the experiments we take $\Delta = 1$ which makes $2F = F_1 = F_2$. The topographic forcing is given by:

\[
\eta_b = \frac{f_0 h_b L}{D_2 U} \quad (5)
\]

and is only included in the lowest layer.
We consider a purely zonal basic state

\[ U_n(y) = -\frac{\partial \Psi_n}{\partial y} \]  

that can exhibit a horizontal shear as a source of kinetic energy and a vertical shear (through thermal wind) as a source of available potential energy. Let \( \phi_n \) be the eddy stream function such that

\[ \psi_n = \Psi_n(y) + \phi_n(x, y, t) \]  

with eddy velocities

\[ u_n = -\frac{\partial \phi_n}{\partial y}, \quad v_n = \frac{\partial \phi_n}{\partial x}. \]  

Substitute Eq. 7 into Eq. 1 to get

\[ \frac{\partial}{\partial t} + U_n \frac{\partial}{\partial x} q_n + \frac{\partial \phi_n}{\partial x} \frac{\partial Q_n}{\partial y} + \left[ \frac{\partial \phi_n}{\partial x} \frac{\partial q_n}{\partial y} - \frac{\partial \phi_n}{\partial y} \frac{\partial q_n}{\partial x} \right] = 0, \]  

where the potential vorticity gradient of the basic state is

\[ \frac{\partial Q_n}{\partial y} = \beta - \frac{\partial^2 U_n}{\partial y^2} - \frac{F}{2} (-1)^n (U_1 - U_2) + \frac{\partial \eta_n}{\partial y} \delta_{n2}, \]  

and the perturbation potential vorticity is given by

\[ q_n = \nabla^2 \phi_n - \frac{F}{2} (-1)^n (\phi_2 - \phi_1). \]  

We reach the linear stability problem by neglecting the terms of order \( O(\phi^2_n) \) and higher to get the linearized QGPV equation, resulting in

\[ \left[ \frac{\partial}{\partial t} + U_n \frac{\partial}{\partial x} \right] q_n + \frac{\partial \phi_n}{\partial x} \frac{\partial Q_n}{\partial y} = 0, \]  

with boundary conditions

\[ \frac{\partial \phi_n}{\partial x} = 0, \quad y = \pm 1, \]  

which ensures that there is no flow into or out of the walls at the boundary.
The necessary conditions for instability in this case are given by Pedlosky (1987):

$$\int_{-1}^{1} dy \sum_{n} \frac{\partial}{\partial t} \frac{|\phi_n|^2}{|U_n-c|^2} \frac{\partial Q_n}{\partial y} = 0,$$

(14)

where $c$ is the phase speed of the wave (as further developed in section 2.2). This equation is derived from the zonally averaged momentum equation. If the flow is unstable, $\frac{\partial}{\partial t} \frac{|\phi_n|^2}{|U_n-c|^2}$ must be positive such that the eddy energy is growing. In order for this equation to be satisfied, $\frac{\partial Q_n}{\partial y}$ must change sign somewhere in the domain. This theorem requires purely zonal basic state flows; we acknowledge that the atmosphere and ocean are decidedly non-zonal, a potential topic for exploration in future work.

\textit{b. Diagnostics}

In order to quantify the energetics of the flow, we use the energy equation for the perturbations. First let $D_n/D = d_n$ then take Eq. 12 and multiply by $-d_n \phi_n$, take a zonal integral (denoted by overbar) and sum the two layers, and integrate in $y$ to yield:

$$\frac{\partial}{\partial t} \int_{-1}^{1} dy \left[ \text{EKE}_1 + \text{EKE}_2 + \text{EAPE} \right] = \int_{-1}^{1} dy \left[ \Delta \text{EKE}_1 + \Delta \text{EKE}_2 + \Delta \text{EAPE} \right],$$

(15)

$$\text{EKE}_n = \frac{d_n}{2} \left( \frac{\partial \phi_n}{\partial x} \right)^2 + \left( \frac{\partial \phi_n}{\partial y} \right)^2,$$

(16)

$$\text{EAPE} = \frac{(\phi_1 - \phi_2)^2}{2} F,$$

(17)

$$\Delta \text{EKE}_n = d_n \frac{\partial \phi_n}{\partial x} \frac{\partial \phi_n}{\partial y} \frac{\partial U_n}{\partial y},$$

(18)

$$\Delta \text{EAPE} = F(U_1 - U_2) \frac{\partial \phi_2}{\partial x} \phi_1.$$  

(19)

The left hand side of Eq. 15 represents the time change of total eddy energy and the right-hand side represents the conversion from the mean into the eddy energy. If there is energy converted into eddies through a positive $\Delta \text{EKE}_n$, then we consider the momentum fluxes to be downgradient, because they act to relax the mean flow and transfer kinetic energy from the mean to the eddies.
(Strictly speaking, the eddy flux is downgradient if $u'v'\partial U/\partial y < 0$, but we are focused on the extraction of energy from the mean flow.) If there is energy converted into eddies through a positive $\Delta$EAPE, then we the buoyancy fluxes to be downgradient because they act to relax the mean buoyancy gradient and transfer potential energy from the mean to the eddies. Conversely, fluxes are said to be upgradient if they have a negative eddy conversion, and they act to transfer energy from the eddies to the mean.

Another way to assess how eddies affect the mean flow is to consider the zonally averaged, zonal momentum equation summed over both layers:

$$\frac{\partial}{\partial t} \left[ \sum_{n=1}^{2} d_n u_n \right] = -\frac{\partial}{\partial y} \left[ \sum_{n=1}^{2} d_n u_n v_n \right], \quad (20)$$

where the change of zonal momentum is related to the divergence of the Reynolds stresses. The potential vorticity equation and the enstrophy equation can be combined to show that

$$\frac{\partial}{\partial t} \left[ \sum_{n=1}^{2} d_n u_n \right] = -\frac{\partial}{\partial t} \left[ \sum_{n=1}^{2} d_n \frac{\overline{q_n^2}}{2\partial Q_n/\partial y} \right]. \quad (21)$$

Ultimately, we want to know if the barotropic flow accelerates or decelerates; this expression tells us the sign of the acceleration if $\partial Q_n/\partial y$ has the same sign in both layers (Held 1975). If the potential vorticity gradients are not the same sign, then the sign of the acceleration also depends on the ratio of the magnitudes of $\overline{q_n^2}$ as well as $\partial Q_n/\partial y$.

c. Eigenvalue Solver

To solve Eq. 12 over a range of conditions, an eigenvalue solver is required. First, assume a plane wave solution of the form

$$\phi_n = \text{Re} \Phi_n(y)e^{i(kx-ct)}, \quad (22)$$

so that Eq. 12 becomes two coupled ordinary differential equations:

$$(U_1 - c) \left[ \frac{d^2 \Phi_1}{dy^2} - k^2 \Phi_1 - \frac{F}{2} (\Phi_1 - \Phi_2) \right] + \Phi_1 \frac{\partial Q_1}{\partial y} = 0 \quad (23)$$

$$(U_2 - c) \left[ \frac{d^2 \Phi_2}{dy^2} - k^2 \Phi_2 - \frac{F}{2} (\Phi_2 - \Phi_1) \right] + \Phi_2 \frac{\partial Q_2}{\partial y} = 0 \quad (24)$$
We discretize the various parameters across the channel with 128 points in Y and use second order finite differencing to create a differentiation matrix for the operator $\frac{d^2}{dy^2}$ which gives an equivalent matrix expression for Equations 23 and 24. Then for every $k$, we use MATLAB’s eigenvalue solver to compute the eigenvalues $c$ and the eigenvectors $\left[ \frac{d^2\Phi_n}{dy^2} - k^2\Phi_n + \frac{F}{2}(-1)^n(\Phi_1 - \Phi_2) \right]$ and $\Phi_n$. For convenience, we normalize $\Phi_n$ so that the total energy in the domain is 1 ($E_{KE_1} + E_{KE_2} + E_{APE} = 1$). Then growth rates $kc_i$ and energy conversions are computed (Eq. 18 and 19). After computing the solution for all $k$’s, the solution $\Phi_n$ with maximum growth rate $kc_i$ from linear analysis is selected for further analysis. In the nonlinear problem, the solution that is selected is not always the one with the maximum growth rate, but can be another mode (Pedlosky 1981). Nevertheless, the linear solution often provides insight into the dynamics of the system.

3. Parameter Sweep Experiments with a Gaussian Jet

First we will consider a case where we have a basic state Gaussian jet and no topography. This base case represents a system that is baroclinically unstable but has a relatively narrow zonal jet. This is a scenario evocative of Gulf Stream or Kuroshio extensions, or narrow jets in the Southern Ocean, but in an idealized, pared-down sense. This is relevant for us to examine.

a. Set Up

We consider the two layer channel with a Gaussian jet of half-width $\delta$ in the upper layer, no flow in the lower layer, and $\eta = 0$ (Fig. 1).

$$U_1 = \frac{1}{2} + \frac{1}{2}e^{-\frac{y^2}{\delta^2}}$$  \hspace{1cm} (25)

$$U_2 = 0$$  \hspace{1cm} (26)

Note that the velocity has been scaled out of the problem and is included in the $\beta$ parameter. $\delta$ ranges from 0 to 1 in our non-dimensionalized domain. In this case the basic state potential
Fig. 1. Example (a) zonal velocity profiles and (b) potential vorticity gradients for $\beta = 60$, $F = 100$ and $\delta = 0.1$ for the jet experiments with the two layers quasi-geostrophic channel model. For this type of experiment, there is necessarily a change of sign of the meridional potential vorticity gradient between the two layers, a necessary condition for baroclinic instability. The parameters are varied so that the potential vorticity gradients change sign in the horizontal as well.

The potential vorticity gradient is given by

\[
\frac{\partial Q_1}{\partial y} = \beta - \frac{\partial^2 U_1}{\partial y^2} + \frac{F}{2} U_1 \tag{27}
\]

\[
\frac{\partial Q_2}{\partial y} = \beta - \frac{F}{2} U_1 \tag{28}
\]

There are 4 non-dimensional parameters to vary in this case: $F$, $\delta$, $\Delta$, and $\beta$. For experiments discussed here, we set the ratio of the layer thicknesses $\Delta = 1$ and vary the others.

b. Results

First consider the energy conversion over a range of $F$, the stratification parameter, and $\beta$, the planetary PV gradient, for a fixed jet width $\delta = 0.05$, to be sufficiently narrow that the jet can go barotropically unstable. We scale the constants $F$ and $\beta$ with $\delta^2$ where $\delta = \delta_0/L$ is the length scale of the jet width. This yields the scaled stratification parameter

\[
F\delta^2 = \frac{f_0^2 \delta_0^2}{g(\rho_2 - \rho_1)/\rho_0 D} \tag{29}
\]
Fig. 2. Energy conversion computed for the most unstable eigenfunctions normalized by total (potential plus kinetic) energy conversion at jet width parameter $\delta = 0.05$ (a narrow jet). Panel (a) shows potential energy conversion and (b) kinetic energy conversion. The domain is divided into 5 regions. The dotted line shows where the momentum fluxes change sign and the solid line shows where the upper layer potential vorticity gradient changes sign, with no sign change to the right and a sign change to the left. Blue represents down-gradient fluxes and red represents upgradient fluxes. Panel (c) shows a cross section of kinetic energy conversion along the red line of panel (b). The black line marks the point where there is a sign change in the horizontal of the meridional potential vorticity gradient; it does not coincide with a reversal of the momentum fluxes. The white regions indicate where the model admits no solution.
and the effective planetary vorticity gradient.

\[ \beta \delta^2 = \beta_0 \frac{\delta_0^2}{U} \]  

(30)

The deformation radius becomes much larger than the length scale defining the horizontal shear \((F \delta^2 << 1)\), making potential vorticity gradients and growth rates dominated by the horizontal shear. When \(F \delta^2 >> 1\) the baroclinic mode dominates. This result is discussed at length in Killworth (1980). In addition, when \(\delta\) becomes larger the jet is larger in the domain. The change in this horizontal length scale compared to the wall locations suggest the possibility of the walls interfering with horizontal modes.

Figure 2 shows the energy conversion, both potential and kinetic over the range of parameters in the system. Firstly, in figure 2(a), the potential energy conversion is always downgradient as constructed, but sometimes significantly weaker as we weaken the coupling between the layers \((F \delta^2)\). When we consider the momentum fluxes (Figure 2(b)), we observe that when the potential vorticity gradient changes sign in the upper layer, as shown by the black line, the momentum fluxes are not always downgradient. There are also downgradient momentum fluxes associated with no sign change of the potential vorticity gradient in the upper layer. Thus, it is not the introduction of a sign change in the upper-layer potential vorticity gradient that leads to downgradient momentum fluxes. As \(F \delta^2\) decreases or the coupling decreases, the kinetic energy conversion comes to dominate over the potential energy conversion (Figure 2).

To quantify different qualitative behaviors, we split the domain into 5 regions. Region 1 exhibits downgradient momentum fluxes, even though there is no sign change in the upper layer potential vorticity gradient, an unexpected result if one naively assumes that the conditions for barotropic instability must be satisfied for the flow to extract energy from the mean KE. Region 2 has downgradient momentum fluxes and a sign change in the upper layer potential vorticity gradient, while region 3 has upgradient momentum fluxes and no sign change in the upper layer potential vorticity gradient. Region 4 has upgradient momentum fluxes and a sign change in the upper layer potential vorticity gradient, a second unexpected result, in that the necessary conditions for barotropic instability are now satisfied, but the flow is unable to extract energy from the mean.
KE. Region 5 also exhibits a sign change in the upper layer potential vorticity gradient, but is now dominated by barotropic conversion.

These 5 different regions exhibit defining characteristics in the wavenumber space (Fig. 3). In Regions 1-4, the dominant mode (with largest growth rate) is a mode with primarily baroclinic production (downgradient buoyancy fluxes) and weak barotropic production (kinetic energy fluxes). When a mode has primarily kinetic energy conversion, we denote it a barotropic mode and when it has primarily baroclinic conversion, a baroclinic mode. Regions 1 and 2 have downgradient kinetic energy fluxes and regions 3 and 4 have upgradient kinetic energy fluxes at the most unstable wavenumber. Comparing region 1 and region 2, the primary difference is that modes with higher wavenumber appear in region 2. These new modes have primarily barotropic conversion. Similarly, these modes appears in region 4 but not region 3. Thus when the sign change is introduced into the upper layer potential vorticity gradient in regions 2 and 4, a higher-wavenumber barotropic mode appears as a solution.

In these regions 2 and 4, the dominant barotropic modes both have locations where $U_1 - c$ vanishes. This indicates that the barotropic modes are contiguous with neutral modes, and is a result consistent with previous studies (Kuo 1949). The specifics of this linear model do not allow for the generation of a critical layer instability, but it is unclear how important this is for the dynamics (Robinson 1974). The dominant (baroclinic) mode sets the direction of the momentum fluxes in regions 1-4 and the mode primarily dependent on the physical parameters $F$ the stratification, $\beta$ the planetary vorticity gradient, and $\delta$ jet width but not on the upper-layer potential vorticity gradient in particular, as evidenced by the momentum fluxes changing sign independent of the upper-layer potential vorticity gradient. In region 5, the baroclinic mode’s growth rate decreases while the barotropic mode’s growth rate increases and becomes dominant. This tells us that when $F$ is reduced, coupling between the layers weakens, associated with an increase in the internal deformation radius. Physically this corresponds to a weakening of the baroclinic mode, largely without modifying the barotropic mode (Fig. 3).

The direction of the momentum fluxes in the baroclinic mode can be visualized geometrically (Fig. 4). When the stream function is tilted with the flow, then the fluxes are upgradient and energy is being fed back into the mean, but when the stream function associated with the most unstable mode is tilted against the flow, the momentum fluxes are downgradient and energy is
Fig. 3. Example growth rates and normalized energy conversions from 5 different regions. Panel (a) shows growth rates, (b,d) kinetic energy conversion, and (c,e) potential energy conversion. Panel (d) shows the same data as (b), but rescaled for detail; the same holds for (e) and (c). Cases were selected from Region 1: $\beta = 120, F = 520$; Region 2: $\beta = 80, F = 360$; Region 3: $\beta = 280, F = 400$; Region 4: $\beta = 200, F = 280$; and Region 5: $\beta = 240, F = 200$. All cases are for $\delta = 0.05$. The growth rates are small in regions with upgradient momentum fluxes.
Fig. 4. The upper level (a) jet profile, a function of y alone, and the most unstable eigenfunctions plotted in x-y space for (b) a case in region 2, $\beta = 20$, $\delta = 0.05$, and $F = 360$ and (c) a case in region 3, $\beta = 320$, $\delta = 0.05$, and $F = 400$ in region 3. The tilt of eigenfunctions at the center of the domain indicate momentum flux direction near the jet core, revealing downgradient momentum fluxes in panel (b) and upgradient fluxes in panel (c).

being extracted from the mean. An example from region 2 reveals that near the center of the jet, the stream function is tilted against the flow leading to downgradient momentum fluxes. (The flux associated with the linear mode does become upgradient on the flanks of the jet.) An example from region 3, on the other hand, shows that the stream function is everywhere tilted with the flow, indicating upgradient momentum fluxes.

When the jet is widened by increasing $\delta$ similar structures appear but the line that separates upgradient and downgradient momentum fluxes moves to the right (larger effective planetary vorticity gradient $\beta \delta^2$), eventually until there is no region 4 (Fig. 5), essentially eliminating the upgradient momentum fluxes.

4. Exploring Symmetries in the System using a Cosine Jet Configuration

In the previous section, we examined cases where there was always a sign change between the two layers permitting baroclinic instability. In this section, we create a second jet configuration, with a basic flow and zonally-uniform topography, to create the reverse conditions: there is always a sign change in the potential vorticity gradient within each of the layers, but only sometimes a sign change in the potential vorticity gradient between the layers. By comparing these two cases,
Fig. 5. The same as Fig. 2, panels (a) and (b), but for \( \delta = 0.3 \). As the jet becomes wider (larger \( \delta \)), the region with upgradient momentum fluxes all but disappears.

we expect a symmetry because baroclinic and barotropic instability are described by the same eigenvalue problem (Drazin and Reid 2004), and can be viewed as rotations of the same system.

The zonally uniform topography is specifically constructed to ensure that the meridional potential vorticity gradient in both layers changes sign at the exact same point, but not necessarily in the vertical (Figure 6). It is not intended to provide insight into oceanic flows (where this contrived alignment would be vanishingly unlikely), but rather to illustrate how the mathematics of baroclinic vs. barotropic instability are the same.

\textit{a. Set Up}

The goal of the cosine jet case is to allow the potential vorticity gradient to be the same sign between the two layers, but a different sign within each layer (Fig. 6). Topography with the exact same shape as the mean flow is introduced in the lower layer in order to force the potential vorticity gradients to change sign at the same location in both layers. This creates a necessarily
Fig. 6. An example (a) velocity profile and (b) potential vorticity gradient for the cosine jet case, with $\beta = 1$, $F = 4$. The topography must have the same exact shape as the upper level jet profile to ensure the meridional potential vorticity gradient reverses sign at the same point in both layers; the gradient does necessarily change in the vertical, however. This is an equivalent-symmetric case to the Gaussian jet set up, except the horizontal gradients become vertical gradients.

barotropically unstable configuration, but not one that is necessarily baroclinically unstable. The flow is then:

\begin{align}
U_1 &= 1/2 + 1/2 \cos(\pi y) \\
U_2 &= 0
\end{align}

with topography as

\begin{equation}
\frac{\partial \eta_b}{\partial y} = \xi \cos(\pi y) + C.
\end{equation}

The potential vorticity gradients are then given by

\begin{align}
\frac{\partial Q_1}{\partial y} &= \beta + \frac{\pi^2}{2} \cos(\pi y) + \frac{F}{4} (1 + \cos(\pi y)) \\
\frac{\partial Q_2}{\partial y} &= \beta - \frac{F}{4} (1 + \cos(\pi y)) + \xi \cos(\pi y) + C
\end{align}
where $\xi = 8$ is the magnitude of the topography variation and $C$ is the magnitude of the constant slope added to enforce the condition that the potential vorticity gradient change sign at the same location in both layers.

**b. Results**

As in section 3, we vary the stratification $F$ and the planetary PV gradient $\beta$, fixing $\xi$ and $C$. The eddy energy conversion from the linear eigensolutions reveals three distinct regions in the domain (Fig. 7). Region 1 has upgradient buoyancy fluxes and downgradient momentum fluxes, with no sign change between the two layers, the signature of barotropic instability. Region 2 exhibits a signature of baroclinic instability, downgradient buoyancy and momentum fluxes, exhibits no sign change between the two layers. This is an unexpected result because it is not baroclinically unstable. Region 3 has downgradient buoyancy fluxes, a signature of baroclinic instability, but has
a sign change between the two layers. Unlike for the momentum fluxes of the Gaussian jet case, there is no region with upgradient buoyancy fluxes and a sign change between the layers.

In transitioning from region 1 and region 2, the buoyancy fluxes change from upgradient to downgradient (Fig. 8). This transition is associated with the barotropic mode of the flow. When the sign change between the two layers is introduced in region 3, a baroclinic mode appears, just as in section 3.2 when a barotropic mode appeared when the horizontal potential vorticity sign gradient was introduced. Thus, primarily barotropic or baroclinic modes appear when the relevant sign change in the potential vorticity gradient appear, but this doesn’t necessarily mean that they are the mode with the dominant growth rate. This highlights the non-seperable nature of this flow because the existence of the instability doesn’t necessarily mean that it sets the direction of the fluxes if the dominant instability is sufficiently dominant.

As $F$ increases, the coupling increases between the two layers and the baroclinic mode becomes stronger and eventually dominates. With a smaller $\beta$, there is a larger region of down-gradient buoyancy fluxes but no sign change. The similarities seen from the parameter sweeps in the Gaussian jet (Figure 2) and the cosine jet (Figure 7), highlight the similarity between barotropic and baroclinic instability, as we switch horizontal and vertical or buoyancy and momentum. The primary difference between the two is the lack of resolution in the vertical because the two layer configuration strongly limits the vertical modes available to the flow.

5. Conclusions and Discussion

We have shown that the momentum flux associated with linear instabilities of a baroclinically unstable flow is very sensitive to the meridional shear. Whether the momentum flux it is upgradient, strengthening the large scale barotropic shear, or downgradient, extracting energy from the kinetic energy of the mean flow, is, for example, not tied to the necessary conditions for barotropic instability, i.e., a sign change in a layer of the meridional potential vorticity gradient. Upgradient momentum fluxes are possible when the flow satisfies the necessary criteria for barotropic instability, and likewise, downgradient momentum fluxes can occur even when the flow remains stable to barotropic instability. Similarly, in a barotropically unstable flow, the direction of the buoyancy fluxes is not determined by whether or not there is a sign change in the meridional potential vorticity gradient in the vertical. This emphasizes the interconnected nature of the buoyancy and
Fig. 8. As in Fig. 3, example growth rates and normalized energy conversions from 3 different regions. Panel (a) shows the growth rates, (b,d) kinetic energy conversion, and (c,e) potential energy conversion. Cases were selected from Region 1: $F = 4, \beta = 1$; Region 2: $F = 7, \beta = 1$; and Region 3: $F = 10, \beta = 1$. The dominant modes switch to higher wavenumbers compared to the Gaussian jet configuration. The KE and PE conversion for regions 1 and 2 are very large and are not shown in the range plotted, chosen to highlight the zero crossing of the energy conversion.
momentum fluxes through the potential vorticity, independent of the naive conditions for baroclinic
or barotropic instability.

As acknowledged in the introduction, there exists a deep body of work on linear instability of shear
flows in the atmosphere and ocean, from fundamental work in the 1940s (e.g., Charney 1947; Eady
1949) and 1970s (e.g., McIntyre 1970; Held 1975; Killworth 1980; Holland and Haidvogel 1980), to
more recent work in the context of barotropic shear’s ability to inhibit baroclinic instability (James
1987; Nakamura 1993a). Our results suggest, however, that further work, even at the fundamental
level, is still needed to fully understand mixed shear instability. While it is ultimately the nonlinear
problem that is most relevant to observed flows, we have shown the range of complexity already
present in the linear problem. Future work could explore eddy life cycle experiments, particularly
in the regions of parameter space where the momentum fluxes appear disconnected from the criteria
for barotropic instability.

Where are these results most relevant to oceanic circulation? While the role of upgradient
momentum fluxes in driving the surface westerlies of the atmosphere is well known, up- and
downgradient fluxes of momentum driven by potential vorticity gradient instabilities may play a
role in the oceanic circulation as well. The Southern Ocean is perhaps the most likely candidate:
Williams et al. (2007) found regions of eddy vorticity flux convergence and divergence in the
Southern Ocean based on altimetry data, indicating the transport of momentum by mesoscale
edds. High resolution model integrations of jets in the Southern Ocean also reveal the potential
for upgradient momentum fluxes like those in the atmosphere (Li et al. 2016). Li et al. (2018)
followed up with evidence from Argo floats of thermally indirect overturning created by these
upgradient momentum fluxes, analogous to the Ferrell cell. Finally, Youngs et al. (2017) found
evidence of downgradient momentum fluxes and barotropic instability near topographic features
in an idealized model of the Southern Ocean. Downstream of the topography, however, upgradient
momentum fluxes appeared.

The Gulf Stream also exhibits mixed barotropic-baroclinic instability, with the potential for
upgradient momentum fluxes in the extension region, as observed in models as far back as Semtner
and Mintz (1977). Williams et al. (2007) also found evidence in the altimetry data that eddy
vorticity fluxes play a leading order in momentum transport in the Gulf Stream and other western
boundary current extension regions, e.g., the Kuroshio (Williams et al. 2007). These extension
regions, where the boundary currents separate from the coast, have been further investigated using high resolution, regional oceanic models (Gula et al. 2015). Finally, there is evidence mixed barotropic-baroclinic instability in the Labrador Sea from satellite altimetry (Zhai et al. 2008).

Low-resolution models generally don’t include a representation of the momentum flux in their parameterizations, effectively assuming it is negligible and focusing only on the buoyancy forcing from unresolved eddies (e.g., Gent and McWilliams 1990). Even current high resolution models likely underestimate the backscatter of kinetic energy from the eddies to the mean (Hewitt et al. 2020). Frontier work in parameterization development seeks to account this backscatter in a multiple ways, for instance, with stochastic modeling (e.g., Grooms and Majda 2013; Mana and Zanna 2014) or using an estimate of available potential energy that is lost, and feeding it back into the momentum equation (e.g., Bachman 2019; Jansen et al. 2019; Juricke et al. 2020), amongst others. There are also efforts to develop data driven parameterizations with machine learning (e.g., Guillaumin and Zanna 2020). An effective parameterization would need to be able to identify regions of downgradient momentum fluxes associated with barotropic dominated instabilities as well. While mixed-shear instability are not the only processes leading to backscatter, e.g., the submesoscales play a role as well (Schubert et al. 2019; Ajayi et al. 2020), it is our hope that a better understanding of the underlying instabilities could aid in the development in new parameterizations.

As highlighted by the symmetry between our Gaussian and cosine jet configurations, the distinction between barotropic and baroclinic flows is somewhat artificial; they are connected through potential vorticity and both buoyancy and momentum fluxes need to be considered if there is horizontal and vertical shear. To highlight this connection, a rotation of the system can flip the perspective from baroclinic to barotropic instability (Drazin and Reid 2004). We have referred to these instabilities as mixed barotropic-baroclinic instability, or mixed shear instability, to highlight connections to conceptual work on both areas. A better, more general term might simply be “potential vorticity wave instability” to reflect that the mechanism is driven by waves on potential vorticity gradients, the sign change in gradient allowing two waves to couple and extract energy from the background flow.

Acknowledgments. Much of this work was done with the advice of Joe Pedlosky. Glenn Flierl provided Matlab code and advice on the manuscript. MKY was supported by the Woods Hole Geophysical Fluid Dynamics Fellowship, the National Defense Science and Engineering Graduate
Fellowship, and the NOAA Climate and Global Change Postdoctoral Fellowship. EPG acknowledges support from the NSF through award OAC-2004572.

Data availability statement. The code that supports the findings of this study is openly available in Zenodo at https://doi.org/10.5281/zenodo.4474545.

References


